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On a direct bilinearization method: Kaup's higher-order water wave equation as a modified nonlocal Boussinesq equation

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Abstract. A systematic procedure for the bilinearization of classes of soliton equations is developed with the help of a generalization of Bell's exponential polynomials. Application of this procedure to Kaup's higher-order wave equation discloses several links with other soliton systems. In particular, it is found that the Kaup equation is the modified version of a sech square soliton system which constitutes an alternative to the good Boussinesq equation.

1. Introduction

Bilinear forms are generally recognized as an efficient tool to prove the existence of N-soliton solutions to nonlinear partial differential equations (NLPDEs) which are presumed to constitute a soliton system, and to relate these equations to Sato's framework of integrable hierarchies (Jimbo and Miwa 1983). They are also useful in disclosing hidden links between different soliton systems. By deriving a bilinear Bäcklund transformation for the Boussinesq equation, Hirota and Satsuma (1977) obtained insight into other soliton equations including a modified Boussinesq equation and Kaup's higher-order water wave equation (Kaup 1975). Experience has given strength to Hirota's claim that every soliton system should be derivable from a fundamental equation (or system of equations) in bilinear form. However, the crucial question is '*How does one find the bilinear equations, or the appropriate dependent variable transformations, without relying on clever guesswork*?' A possible answer may be obtained by relating the Hirota procedure to other direct techniques, such as similarity methods (Ludlow and Clarkson 1993) or the Painlevé singularity analysis (Gibbon *et al* 1985, Hietarinta and Kruskal 1992). It may therefore be important to identify the algebraic ingredients which are common to these methods, and to examine how simple partition polynomials, related to the definition of the Hirota D-operator, may lead to the bilinearizing transformations.

In this paper we propose a direct bilinearization scheme, based on a notion of scale invariance and on the use of a generalization of Bell's exponential polynomials (Bell 1934). We show how these polynomials may be used to decide whether a given NLPDE may be derived from a single bilinear equation of KdV type (Hietarinta 1987), or from a similar two field system. As we bilinearize the Kaup equation we obtain the 'Miura' link which relates this equation to a sech squared soliton system—we call it a nonlocal alternative to the Boussinesq equation (NLBq)—and to an equivalent real form (Boiti *et al* 1981) of the nonlinear Schrödinger (RNLS) equation. The Kaup equation and the RNLS equation are

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hereby found to inherit the solutions of a linearizable equation of the (potential) Burgers type.

It may be noticed that the relation between the Kaup equation and the nonlinear Schrödinger equation has been established by Hirota (1985), and that the NLBq system corresponds to the integrability condition of a linear system discussed by Hirota and Satsuma (1977). Yet, the important point is that the above connections are obtained as a result of a straightforward procedure in which all transformations are accounted for. Furthermore, it is worth specifying the form of the sech squared soliton system with Boussinesq dispersion of which Kaup's equation is the modified partner, and which, unlike the Boussinesq equation, cannot be derived from a single bilinear equation of the KdV type.

2. Bell polynomials

As a background for discussing the Hirota method we consider the Burgers–Hopf hierarchy (Choodnovsky 1977) of NLPDEs which can be linearized by means of the Cole–Hopf transformation. To express the link between these linearizable equations and the simplest types of bilinearizable equations it is useful to consider special families of partition polynomials.

Consider a C^∞ function $q(x)$, the variable $q_r = \partial_x^r q$, $r = 1, 2, \ldots$, and Bell's exponential polynomials:

$$Y_{nx}(q) \equiv Y_n(q_1, \ldots q_n) = \mathrm{e}^{-q}\partial_x^n \mathrm{e}^q = \sum \frac{n!}{c_1! \ldots c_n!(1!)^{c_1} \ldots (n!)^{c_n}} q_1^{c_1} \ldots q_n^{q_n}. \tag{1}$$

where the sum is to be taken over all partitions of $n : n = c_1 + 2c_2 + \cdots + nc_n$. Thus:

$$Y_x(q) = q_x \qquad Y_{2x}(q) = q_{2x} + q_x^2 \qquad Y_{3x}(q) = q_{3x} + 3q_x q_{2x} + q_x^3 \qquad \ldots. \tag{2}$$

By introducing several independent variables it is a straightforward matter to extend the Bell polynomials to more dimensions. We consider, in particular, the two-dimensional extension:

$$Y_{mx,nt}(q) \equiv Y_{m,n}(q_{r,s}) = \mathrm{e}^{-q}\partial_x^m \partial_t^n \mathrm{e}^q \qquad q_{r,s} = \partial_x^r \partial_t^s q(x,t) \tag{3}$$

which still displays a simple partitional structure (each term corresponds to a partition of the available derivatives and carries a coefficient equal to the combinatorial weight of that partition):

$$Y_{x,t}(q) = q_{x,t} + q_x q_t \qquad Y_{2x,t}(q) = q_{2x,t} + q_{2x}q_t + 2q_x q_{x,t} + q_x^2 q_t \qquad \ldots. \tag{4}$$

The link between Y-polynomials and the Burgers–Hopf hierarchy follows from the definition (3) according to which:

$$Y_{mx,nt}(q = \ln\psi) = \psi^{-1}\psi_{mx,nt}. \tag{5}$$

Hence, it is clear that any NLPDE which is a linear combination of Y-polynomials can be linearized by the transformation $q = \ln\psi$.

These Y-polynomials can therefore be used as a guide to decide whether a given NLPDE can be linearized by this transformation or by a closely related one, such as the Cole–Hopf transformation. First one must look for the dimensional constraints on the variables (dependent and independent) which are imposed by the requirement that the equation be invariant under a scale transformation. One then rewrites the NLPDE in terms of a 'primary field' which is introduced as a dimensionless alternative to the original field and which is multiplied by a dimensionless (free) scaling constant. The primary field and the value of the constant should be chosen in such a way that the equation can the expressed as a linear combination of Y-polynomials, up to possible overall differentiations. As a simple example we consider the Burgers equation:

$$w_t - w_{xx} + \alpha w w_x = 0 \tag{6}$$

which is invariant under the scale transformation:

$$x \to \lambda x \qquad t \to \lambda^r t \qquad w \to \lambda^d w \tag{7}$$

if $d - r = d - 2 = 2d - 1$ or $r = 2$ and $d = -1$ (we assign a dimension 1 to x, and consider α as a dimensionless parameter). A dimensionless primary field $q(x, t)$ is easily introduced by setting $w = cq_x$, with $c =$ dimensionless constant. Equation (6) then becomes

$$\left(q_t - q_{2x} + \frac{c\alpha}{2} q_x^2\right)_x \equiv [Y_t(q) - Y_{2x}(q)]_x + (2 + c\alpha) q_x q_{2x} = 0. \tag{8}$$

The obvious reason for expressing the left-hand side of (6) as a derivative is that a quadratic nonlinearity excludes the use of Y-polynomials of orders larger than 2. Equation (8) tells us that c should be taken equal to $-2/\alpha$, and that (6) is linearized by application of the Cole–Hopf transformation

$$w = -\frac{2}{\alpha} \partial_x \ln \psi.$$

3. Binary Bell polynomials

Setting $F = e^f$ and $G = e^g$, it follows from the definition of the Hirota operators

$$D_x^m D_t^n F \cdot G = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n F(x, t) G(x', t')|_{x'=x, t'=t} \tag{9}$$

that

$$\begin{aligned} (FG)^{-1} D_x^m D_t^n F \cdot G &= \sum_{p=0}^{m} \sum_{q=0}^{n} (-)^{p+q} \binom{m}{p} \binom{n}{q} Y_{m-p, n-q}(f_{r,s}) Y_{p,q}(g_{r,s}) \\ &= Y_{m,n}(y_{r,s} = f_{r,s} + (-)^{r+s} g_{r,s}) \end{aligned} \tag{10}$$

where we have used the addition property

$$Y_{mx,nt}(f + g) = \sum_{p=0}^{m} \sum_{q=0}^{n} \binom{m}{p} \binom{n}{q} Y_{(m-p)x,(n-q)t}(f) Y_{px,qt}(g) \tag{11}$$

and the parity property

$$Y_{m,n}[(-)^{r+s} q_{r,s}] = (-)^{m+n} Y_{m,n}(q_{r,s}) \tag{12}$$

which follow straight away from definition (3) and from the explicit representation

$$Y_{m,n}(q_{r,s}) = \sum \frac{m!n!}{c_1!\dots c_k!} \prod_{j=1}^{k} \left(\frac{q_{r_{1j}}, q_{r_{2j}}}{r_{1j}!r_{2j}!} \right)^{c_j} \tag{13}$$

where the sum is to be taken over all partitions $[(r_{11}, r_{21})^{c_1}, \dots (r_{1k}, r_{2k})^{c_k}]$ of (m, n).

Formula (10) suggests that one should introduce 'binary' Bell polynomials which are defined in terms of partial derivatives of two dimensionless field variables $u(x, t)$ and $v(x, t)$, and which obey the partitional recipe of the Y-polynomials combined with a simple parity rule:

$$\mathcal{Y}_{mx,nt}(u, v) = Y_{mx,nt}(y)\Big|_{y_{r,s} = \begin{cases} u_{r,s} & \text{if } r+s \text{ is even} \\ v_{r,s} & \text{if } r+s \text{ is odd} \end{cases}} \tag{14}$$

i.e.

$$\begin{aligned} &\mathcal{Y}_{x,0}(u, v) \equiv \mathcal{Y}_x(v) = v_x \qquad \mathcal{Y}_{0,t}(u, v) \equiv \mathcal{Y}_t(v) = v_t \\ &\mathcal{Y}_{2x,0}(u, v) = u_{2x} + v_x^2 \qquad \mathcal{Y}_{x,t}(u, v) = u_{x,t} + v_x v_t \\ &\mathcal{Y}_{3x,0}(u, v) = v_{3x} + 3u_{2x}v_x + v_x^3 \qquad \dots \end{aligned} \tag{15}$$

Formula (10) then becomes

$$\mathcal{Y}_{mx,nt}(u = \ln FG,\ v = \ln F/G) = (FG)^{-1} D_x^m D_t^n F \cdot G. \tag{16}$$

It is easy to verify the separability property

$$\mathcal{Y}_{mx,nt}(u, v) = \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r} \binom{m}{s} P_{rx,st}(u - v) Y_{(m-r)x,(n-s)t}(v) \tag{17}$$

with

$$P_{mx,nt}(q) \equiv \mathcal{Y}_{mx,nt}(u = q,\ v = 0) \tag{18}$$

on account of which it is found that the linearizing logarithmic transformation (5) produces the relation

$$\mathcal{Y}_{mx,nt}(u = v + q, v)|_{v=\ln\psi} = \psi^{-1} \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r} \binom{n}{s} P_{rx,st}(q) \psi_{(m-r)x,(n-s)t}. \tag{19}$$

It should be noted that the only non-vanishing contributions to the right-hand side of (19) are those with $r + s =$ even. The even-order P-polynomials contain only even-order derivatives and obey the partitional recipe (13), restricted to even part partitions:

$$P_{2x}(q) = q_{2x} \qquad P_{x,t}(q) = q_{x,t} \qquad P_{4x}(q) = q_{4x} + 3q_{2x}^2 \qquad \dots \tag{20}$$

4. Generating equations and systems

The $\mathcal{Y}$-polynomials can be used to develop a direct bilinearization method, i.e. a procedure for constructing generating equations (or systems of equations) which are quadratic in the dependent variables, and in which all derivatives are expressed through the Hirota D-operator. The term 'generating' means that their solutions should, by construction, generate solutions to the original equation.

We first remark that the formulas (16) and (18) imply

$$P_{mx,nt}(q) = F^{-2} D_x^m D_t^n F \cdot F|_{F=e^{q/2}}. \tag{21}$$

Hence it is clear that NLPDEs which are linear combinations of P-polynomials can be transformed into a single field bilinear equation of the form:

$$\mathcal{F}(D_x, D_t) F \cdot F = 0 \qquad \mathcal{F}:\ \text{polynomial} \tag{22}$$

by the transformation $q = 2 \ln F$.

P-polynomials can therefore be used to decide whether a given NLPDE can be derived from a single bilinear equation of KdV type. Simple examples are the KdV and Bousinesq equations:

$$w_t + w_{3x} + \alpha w w_x = 0 \qquad w_{2t} - w_{4x} + \alpha (w w_x)_x = 0. \tag{23}$$

Invariance under the transformation (7), with $\dim \alpha = 0$ imposes in both cases the constraint $\dim w = -2$. We therefore introduce a dimensionless $q(x,t)$ by setting $w = c q_{2x}$ and rewriting (23) as (the quadratic nonlinearity excludes the use of $P_{mx}(q)$ with $m > 4$)

$$\left[q_{x,t} + q_{4x} + \frac{c\alpha}{2}(q_{2x})^2\right]_x \equiv \left[P_{x,t}(q) + P_{4x}(q) + \left(\frac{c\alpha}{2} - 3\right)(q_{2x})^2\right]_x = 0 \tag{24}$$

$$\left[q_{2t} - q_{4x} + \frac{c\alpha}{2}(q_{2x})^2\right]_{xx} \equiv \left[P_{2t}(q) - P_{4x}(q) + \left(\frac{c\alpha}{2} + 3\right)(q_{2x})^2\right]_{xx} = 0. \tag{25}$$

Choosing, respectively, $c = 6/\alpha$ and $c = -6/\alpha$ we find the generating (primary KdV and primary Boussinesq) equations:

$$q_{x,t} + q_{4x} + 3q_{2x}^2 \equiv P_{x,t}(q) + P_{4x}(q) \equiv e^{-q}(D_x D_t + D_x^4) e^{q/2} \cdot e^{q/2} = 0 \tag{26}$$

$$q_{2t} - q_{4x} - 3q_{2x}^2 \equiv P_{2t}(q) - P_{4x}(q) \equiv e^{-q}(D_t^2 - D_x^4) e^{q/2} \cdot e^{q/2} = 0 \tag{27}$$

and the bilinearizing transformations

$$w = \pm \frac{12}{\alpha} (\ln F)_{xx}.$$

More challenging is the bilinearization of Kaup's higher-order water wave equation (Kaup 1975):

$$K(w) \equiv w_{2t} - \alpha w_{2x} + w_{4x} + \tfrac{1}{2}(w_x^2)_t + (w_x w_t + \tfrac{1}{2} w_x^3)_x = 0 \tag{28}$$

which is closely related to the classical Boussinesq (ClBq) equation (Kawamoto 1984, Hirota 1985).

Some authors (Sachs 1988) find it convenient to transform this equation into a corresponding one with $\alpha = 0$ by the transformation $w \to w + \alpha t$. In the present context it is preferable to bilinearize Kaup's equation as it stands. The importance of leaving the parameter α in the equation will become clear in the following. At this stage it suffices to remark that the process of bilinearization is not affected by the value of α.

Scale invariance of (28) can only be preserved if $\dim w = 0$ (with $\dim\alpha$ equal -2 if $\alpha \neq 0$). Setting $w = cv$ we notice that $K(cv)$ cannot be expressed as a linear combination of P-polynomials (or Y-polynomials), nor as a derivative of such a combination.

Yet, we may try to rewrite $K(cv)$ such as to incorporate as many terms as possible into linear combinations of $\mathcal{Y}$-polynomials, or into derivatives of such combinations. We therefore rewrite the scaled equation (28) in terms of brackets which contain only odd-order derivatives of $v(x,t)$:

$$\left(v_t + \frac{c}{2}v_x^2\right)_t + \left(cv_x v_t + \frac{c^2}{2}v_x^3 + v_{3x} - \alpha v_x\right)_x = 0. \tag{29}$$

The first bracket can be expressed as

$$\left[\mathcal{Y}_t(v) + \frac{c}{2}\mathcal{Y}_{2x}(u,v)\right]_t - \frac{c}{2}u_{2x,t}. \tag{30}$$

The remainder can be incorporated into the second bracket, which then becomes

$$\left[\mathcal{Y}_{3x}(u,v) - \alpha\mathcal{Y}_x(v) - \frac{c}{2}\mathcal{Y}_{x,t}(u,v) - 3\mathcal{Y}_x(v)\left[u_{2x} - \frac{1}{3}\left(\frac{c^2}{2} - 1\right)v_x^2 - \frac{c}{2}v_t\right]\right]_x. \tag{31}$$

By choosing $c = -2\mathrm{i}$ we may express the three remaining terms as a linear combination of $\mathcal{Y}_{2x}(u,v)$ and $\mathcal{Y}_t(v)$. Thus, setting $w = -2\mathrm{i}v$ in (28) we obtain a 'primary' version of the Kaup equation:

$$PK(v) \equiv v_{2t} - \alpha v_{2x} + v_{4x} - 2\mathrm{i}(v_t v_{2x} + 2v_x v_{xt}) - 2(v_x^3)_x = 0 \tag{32}$$

which may be rewritten in the form:

$$[\mathcal{Y}_t(v) - \mathrm{i}\mathcal{Y}_{2x}(u,v)]_t + [\mathcal{Y}_{3x}(u,v) - \alpha\mathcal{Y}_x(v) + \mathrm{i}\mathcal{Y}_{x,t}(u,v) - 3\mathrm{i}\mathcal{Y}_x(v)[\mathcal{Y}_t(v) - \mathrm{i}\mathcal{Y}_{2x}(u,v)]]_x = 0. \tag{33}$$

Taking advantage of the freedom which was introduced with the variable $u(x,t)$ we impose a constraint on this auxiliary field so as to obtain a generating system for equation (32) which involves only linear combinations of $\mathcal{Y}$-polynomials. An obvious choice is the condition: $\mathcal{Y}_t(v) - \mathrm{i}\mathcal{Y}_{2x}(u,v) = 0$. It produces the system:

$$\begin{cases} \mathcal{Y}_t(v) - \mathrm{i}\mathcal{Y}_{2x}(u,v) = 0 \\ \mathcal{Y}_{3x}(u,v) - \alpha\mathcal{Y}_x(v) + \mathrm{i}\mathcal{Y}_{x,t}(u,v) = 0 \end{cases}$$

$$\iff \begin{cases} u_{2x} + v_x^2 + \mathrm{i}v_t = 0 \\ v_{3x} + 3u_{2x}v_x + v_x^3 + \mathrm{i}(u_{x,t} + v_x v_t) - \alpha v_x = 0 \end{cases} \tag{34}$$

which can be expressed in the bilinear form:

$$\begin{cases} (D_x^2 + \mathrm{i}D_t)F \cdot G = 0 \\ (D_x^3 + \mathrm{i}D_x D_t - \alpha D_x)F \cdot G = 0 \end{cases} \tag{35}$$

by application of the formula (16).

The system (35) was obtained in Hirota (1985). Differentiation of the generating system (34) with respect to x produces a system for the fields u_{2x} and v_x which can be identified with the CIBq system discussed in Hirota's paper. Here, we obtain the two-field system (34) straight away from the Kaup equation (the converse is much easier).

According to the formulas (17) and (19) we set $u = v+q$ in order to obtain an equivalent version to the system (34):

$$\left[\begin{array}{l} Y_{2x}(v) + \mathrm{i}Y_t(v) + P_{2x}(q) = 0 \\ Y_{3x}(v) + \mathrm{i}Y_{x,t}(v) + 3P_{2x}(q)Y_x(v) + \mathrm{i}P_{x,t}(q) - \alpha Y_x(v) = 0 \end{array}\right. \tag{36}$$

which can be linearized through the logarithmic transformation $v = \ln\psi$ and recombined as

$$\left[\begin{array}{l} L_1\psi \equiv \mathrm{i}\psi_t + \psi_{2x} + q_{2x}\psi = 0 \\ L_2\psi \equiv (2q_{2x} - \alpha)\psi_x + (\mathrm{i}q_{x,t} - q_{3x})\psi = 0 \end{array}\right. . \tag{37}$$

Notice that the formulas (35) and (37) are related by the transformation: $\psi = F/G$, $q = 2\ln G$.

The system (37) is similar to a coupled linear system discussed by Hirota and Satsuma (1977) and Hirota (1985). It should, by construction, generate solutions to the primary Kaup equation (32). Its compatibility is subject to the constraint $[L_1, L_2]\psi = 0$, which is satisfied if $q(x,t)$ satisfies the NLPDE:

$$\left(q_{2t} - \alpha q_{2x} + q_{4x} + q_{2x}^2 - \frac{q_{x,t}^2 + q_{3x}^2}{q_{2x} - \alpha/2}\right)_x = 0 \tag{38}$$

where we exclude the case $q_{2x} \equiv \alpha/2$, $q_{x,t} \equiv 0$.

If $q_{2x} \equiv \alpha/2$ and $q_{x,t} \equiv 0$ it is obvious that the system (37) reduces to a single equation

$$\mathrm{i}\psi_t + \psi_{2x} + \frac{\alpha}{2}\psi = 0 \tag{39}$$

the solutions of which generate solutions $v = \ln\psi$ to equation (32). Thus, it is found that the (primary) Kaup equation must inherit all the solutions of the equation

$$\mathrm{i}v_t + v_{2x} + v_x^2 + \frac{\alpha}{2} = 0. \tag{40}$$

In fact, it is easy to verify that

$$PK(v) \equiv (-\mathrm{i}\partial_t + \partial_x^2 - 2v_x\partial_x - 2v_{2x})\left(\mathrm{i}v_t + v_{2x} + v_x^2 + \frac{\alpha}{2}\right). \tag{41}$$

5. Kaup's equation as a 'modified' sech squared soliton system

It is clear that by setting $q_{2x} = U$ and $q_{x,t} = V$, equation (38) can be transformed into the first-order system

$$\left[\begin{array}{l} U_t = V_x \\ V_t = \alpha U_x - U_{3x} - (U^2)_x + \left(\frac{V^2+U_x^2}{U-\alpha/2}\right)_x \end{array}\right. . \tag{42}$$

The particular link which relates this system to the Kaup equation is revealed by the system (36) or its equivalent:

$$\begin{bmatrix} U \equiv q_{2x} = F_1(v) \equiv -(v_{2x} + v_x^2) - \mathrm{i}v_t \\ V \equiv q_{x,t} = F_2(v) \equiv -2\mathrm{i}v_x^3 + 2v_x v_t + \mathrm{i}v_{3x} - v_{xt} - \mathrm{i}\alpha v_x \end{bmatrix} \tag{43}$$

which, as a Miura transformation (Hirota and Satsuma 1977), maps the solutions of (32) into solutions of the system (42):

$$\left(q_{2t} - \alpha q_{2x} + q_{4x} + q_{2x}^2 - \frac{q_{x,t}^2 + q_{3x}^2}{q_{2x} - \alpha/2} \right)_x \Bigg|_{\substack{q_{2x}=F_1(v) \\ q_{x,t}=F_2(v)}} \equiv (2v_x - \partial_x) P K(v). \tag{44}$$

In the following we restrict our discussion of the system (42) to the cases $\alpha \geqslant 0$.

(i) Case $\alpha > 0$. When $\alpha > 0$ the system (42) is found to admit sech squared soliton solutions and corresponding singular cosech squared solutions:

$$\begin{aligned} &U_{\mathrm{sol}} = \tfrac{1}{2}k^2 \mathrm{sech}^2\left(\frac{\theta}{2}\right) \qquad U_{\mathrm{sing}} = -\tfrac{1}{2}k^2 \,\mathrm{cosech}^2\left(\frac{\theta}{2}\right) \\ &\theta = -kx + \omega t + \tau \qquad \omega = \pm k\sqrt{\alpha - k^2} \qquad k^2 < \alpha \end{aligned} \tag{45}$$

which, as solitary wave solutions, are identical to those of the Boussinesq type equation (or system):

$$U_{2t} - \alpha U_{2x} + U_{4x} + 3(U^2)_{2x} = 0 \Longleftrightarrow \begin{bmatrix} U_t = V_x \\ V_t = \alpha U_x - U_{3x} - 3(U^2)_x \end{bmatrix}. \tag{46}$$

The first equation in (46) transforms into the 'good' Boussinesq (GBq) equation studied by Manoranjan *et al* (1988) under the rescaling $U \to -U/3$.

The system (42) admits also regular two-soliton solutions:

$$\begin{aligned} &U_2 = 2\partial_x^2 \ln[1 + \exp\theta_1 + \exp\theta_2 + A_{12}\exp(\theta_1 + \theta_2)] \qquad \theta_i = -k_i x + \omega_i t + \tau_i \\ &\omega_i = \epsilon_i k_i (\alpha - k_i^2)^{1/2} \qquad k_i^2 < \alpha \qquad \epsilon_i = \pm 1 \\ &A_{12} = (k_1 + k_2)^{-2}\left(k_1^2 + k_2^2 + \frac{2}{\alpha}k_1^2 k_2^2 - \frac{2}{\alpha}\omega_1\omega_2\right) \end{aligned} \tag{47}$$

which, unlike the GBq two-soliton solutions (Tajiri and Nishitani 1982, Manoranjan *et al* 1988), describe elastic two-soliton collisions for all values of the soliton parameters $0 < k_1 < k_2 < \sqrt{\alpha}$. In fact, it is a straightforward matter to verify that, in contrast with the GBq case, the phase-shift parameter A_{12} remains positive and finite for all such values. The asymptotic decomposition of the expression (47) is therefore standard, the positions (shifts) of the emerging solitons being determined by the direction of propagation of the fastest one (sign of ϵ_1):

$$\begin{aligned} &U_2 \underset{\epsilon_1 t \to \pm\infty}{\longrightarrow} \tfrac{1}{2}k_1^2 \mathrm{sech}^2\left(\frac{k_1 x - \omega_1 t - \tau_1^{\pm}}{2}\right) + \tfrac{1}{2}k_2^2 \mathrm{sech}^2\left(\frac{k_2 x - \omega_2 t + \tau_2^{\pm}}{2}\right) \\ &\tau_1^+ = \tau_1 \qquad \tau_1^- = \tau_1 + \ln A_{12} \qquad \tau_2^+ = \tau_2 + \ln A_{12} \qquad \tau_2^- = \tau_2. \end{aligned} \tag{48}$$

The system (42) with $\alpha > 0$ may thus be regarded as an alternative to equation (46), with solitons of the same functional form but without the soliton resonances observed for equation (46) (Lambert *et al* 1987) and the related instability of high-amplitude solitons (Fal'kovich *et al* 1983 and Manoranjan *et al* 1988).

It may be noticed that the above solutions correspond to solutions $q(x, t)$ of the 'potential' equation (a corresponding equation for $U = q_{2x}$ would clearly be nonlocal):

$$q_{2t} - \alpha q_{2x} + q_{4x} + 3q_{2x}^2 - \frac{2}{\alpha}(q_{2x}q_{2t} + q_{2x}q_{4x} - q_{3x}^2 - q_{x,t}^2 + q_{2x}^3) = 0 \quad (49)$$

which are such that when q_{2x} equals $\alpha/2$ the functions q_{3x} and $q_{x,t}$ must vanish simultaneously. As a result of this property it will turn out that the amplitude of the above two-soliton solutions is bounded by $\alpha/2$.

It is also instructive to remark that (38) may be regarded as the compatibility condition for a *pair* of conjugate systems (35), or (37), which are related by the transformation $t \to -t$. The explicit invariance of the bilinear system (35) under the transformation $(t \to -t, F \rightleftharpoons G)$, which reflects the invariance of (32) under the transformation $(t \to -t, v \to -v)$, implies the existence of a 'Darboux property' between the conjugate linear systems which allows the construction of non-trivial solutions to equation (38) out of trivial ones. Thus, let $q(x, t)$ be a solution of (38) and let ψ be a (non-vanishing) solution of the system (37) for that $q(x, t)$. It follows from the relations $\psi = F/G$ and $q = 2\ln G$ that $\{F = \psi \exp(q/2), G = \exp(q/2)\}$ must solve the bilinear system (35), and that $\{\tilde{F} = \exp(q/2), \tilde{G} = \psi \exp(q/2)\}$ must solve its conjugate. This implies that $\tilde{\psi} = \tilde{F}/\tilde{G} = \psi^{-1}$ must satisfy the conjugate of system (37) in which $q(x, t)$ has been replaced by

$$\tilde{q} = 2\ln\tilde{G} = q + 2\ln\psi. \quad (50)$$

It follows that $\tilde{q}(x, t)$ must also solve equation (38).

Setting $\alpha = 1$ for definiteness, we may use this Darboux property to obtain the following two parameter family of solitary wave solutions $\tilde{U}(x, t; k, \omega)$ to the system (42) out of the trivial solution $U = \frac{1}{2}$, $V = 0$:

$$\tilde{U} \equiv \tilde{q}_{2x} = \tfrac{1}{2} + \tfrac{1}{2}k^2 \operatorname{sech}^2\left(\frac{kx + \omega t}{2}\right). \quad (51)$$

(ii) Case $\alpha = 0$. It is easy to see that equation (38) with $\alpha = 0$ must inherit all the solutions of the linearizable equation

$$\mathrm{i}q_t + q_{2x} + \tfrac{1}{2}q_x^2 = 0. \quad (52)$$

This follows from the above Darboux property and from the fact that, at $\alpha = 0$ and $q(x, t) = 0$, the system (37) reduces to the generating equation

$$\mathrm{i}\psi_t + \psi_{2x} = 0 \quad (53)$$

which transforms into equation (52) by the map $q = 2\ln\psi$.

Setting $q_x = w$ into the left-hand side of (38) one also verifies that

$$\begin{aligned}(w_{2t} + w_{4x} + 2w_x w_{2x}) - w_x^{-2} W(w_x, w_t^2 + w_{2x}^2) \\ = [-(\mathrm{i}\partial_t - \partial_x^2) + w_x^{-1}(\mathrm{i}w_t - w_{2x})\partial_x + w_x^{-2} W(w_x, \mathrm{i}w_t - w_{2x})](\mathrm{i}w_t + w_{2x} + w w_x)\end{aligned} \quad (54)$$

where $W(f, g) = fg_x - f_x g$.

When $\alpha = 0$ the system (42) can also be seen to be closely related to the cubic nonlinear Schrödinger (NLS) equation (Hirota and Satsuma 1977, Hirota 1985). It should also be mentioned that equation (38) with $\alpha = 0$ may be identified with a real equivalent to the NLS equation, for a field proportional to q_x, which was derived by Boiti *et al* (1981).

As we conclude, it is worth emphasizing that all the above results have been obtained from a straightforward bilinearization procedure in which all transformations are accounted for.

Let us also remark that, unlike the 'primary' KdV and Boussinesq equations (26), (27), equation (49) does not correspond to a standard bilinear single field equation. It follows that, unlike the modified KdV and modified Boussinesq equations (Hirota and Satsuma 1977), the Kaup equation could not have been obtained by direct application of the bilinear Bäcklund techniques.

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